Computers and Mathematics with Applications 144 (2023) 150–163

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Stability analysis of difference-Legendre spectral method for two-dimensional Riesz space distributed-order diffusion-wave model

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ARTICLE INFO

Keywords:
Difference method
Distributed-order fractional derivative
Stability analysis
Riesz space fractional derivative

ABSTRACT

In this manuscript, a numerical method with high accuracy and efficiency based on the difference-Legendre spectral method is proposed for obtaining the numerical solutions of the two-dimensional space-time distributed-order fractional diffusion-wave equations with the Riesz space fractional derivative. The difference method by an approximate formula respect to the time variable is used to discrete the distribution-order integral part consisting of Caputo fractional derivative. Also, the Gauss quadrature formula respect to the space variable is used to estimate the distribution-order integral part consisting of Riesz space derivative. Further, the stability and convergence analyses are studied for the numerical estimation. Some numerical examples are displayed to show the effectiveness of proposed methods.

1. Introduction

Differential equations of fractional order as an extension and generalization of differential equations of integer order have received a lot of attention among authors. This great attention of authors to fractional differential equations is due to the use of these types of equations to model physics [16,39] and engineering equations [20,22,44]. To describe the phenomena and behaviors of equations with the lacking the temporal or spatial scaling in their natures, we need that the order of the fractional derivative is not single. To compensate for this problem, distributed-order fractional equations become noticeable and important. For the first time, the fractional equation of distributed-order type was studied by Caputo [7] to depict and explain the anomalous processes.

Nowadays, many interests of differential equations of the distributed-order type are used in the field of physics as relaxation phenomena, signal processing and system control. For example in [6] fractional model for description of the relaxation phenomena and anomalous diffusion, and in [9] authors studied the fractional model for regularity on the underlying solutions in time. In order to know the literature of distributed-order fractional derivatives, we recall in [19], authors presented the solutions of 1D fractional diffusion-wave problem of distributed-order by applying the Fourier-Laplace transform and the interpolation scheme. In [24] studied the analytical solutions of time-fractional equation of distributed-order consisting of Caputo fractional operator. In this sense, various numerical methods are worth using to solve fractional equations of distributed-order, for example, finite element method [13], Chebyshev collocation method [28], Petrov-Galerkin and spectral collocation methods [23], local discontinuous Galerkin method [30], finite volume method [25,26], unstructured mesh finite element method [36], fast second-order time two-mesh mixed finite element method [41], a unified Petrov-Galerkin Spectral Method [35], shifted Legendre operational matrix method with Tau method [11], Laplacian operator in axisymmetric cylindrical geometry [1], fast and efficient finite difference method [46], a Galerkin meshless reproducing kernel particle method [3], high-order difference methods [18], operational matrix method [10], hybrid methods [29], finite difference and finite element methods [5], Putzer's algorithm [2], Crank-Nicolson ADI Galerkin-Legendre spectral method [45], a fast implicit difference [21] and compact difference method [42].

The study of the distributed-order fractional calculus is generally an important and significant approach for the modeling of complex systems, decelerating anomalous diffusion and ultra-slow diffusive processes [6,31,38]. It distributes the fractional operators of constant order by integrating

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https://doi.org/10.1016/j.camwa.2023.05.035
Received 28 September 2022; Received in revised form 14 May 2023; Accepted 26 May 2023

the fractional kernel of these operators over an extended range of orders. The distributed-order fractional operator for orders, not great than 2, is displayed by

$$D^{\alpha}=\int_{a}^{b} c(\beta)\frac{d^{\beta}}{dt^{\beta}}d\beta,\ 0\le a<b\le 2,\ c(\beta)\ge 0,$$

in which $\frac{d^{\beta}}{dt^{\beta}}$ is the fractional operator of single-order β and $c(\beta)$ is a non-negative weight function. In particular cases, this integral can be considered as the multi-term fractional derivative over the physical domain. In view of variations of the non-negative weight function, different models of fractional equations can be studied and analyzed in physical phenomena. As an example, we can refer to diffusion, sub-diffusion and fractional relaxation models [31–33], study of composite materials [8] and viscoelastic materials having spatially varying properties [27]. Moreover, some interesting models involving multi-term distributed-order fractional operators were introduced and studied in literature [2,39,45]. Among these studies, the fractional moments (particularly the second moment in 1D) as fundamental criteria

$$\mathbf{M}_{\gamma}(t)=\int_{0}^{\infty} x^{\gamma}u(x,t)dx, \tag{1}$$

are considered for the acceleration and velocity of models. The asymptotic behaviors of fractional moments can be also studied via the Tauberian theorems for specific choices of the density functions [14,39]. Here, in the class of the multi-term distributed-order fractional equations, this paper studies an efficient numerical method for the numerical solution of the two-dimensional space-time distributed-order fractional diffusion-wave equations with the Riesz space fractional derivative

$$\int_{1}^{2} b(\alpha)^{C}\mathfrak{D}_{t}^{\alpha}w(x,y,t)d\alpha=\int_{1}^{2} Q_{1}(\gamma)\frac{\partial^{\gamma}w(x,y,t)}{\partial|x|^{\gamma}}d\gamma+\int_{1}^{2} Q_{2}(\beta)\frac{\partial^{\beta}w(x,y,t)}{\partial|y|^{\beta}}d\beta+\int_{0}^{1} Q_{3}(\mu)\frac{\partial^{\mu}w(x,y,t)}{\partial|x|^{\mu}}d\mu+\int_{0}^{1} Q_{4}(\nu)\frac{\partial^{\nu}w(x,y,t)}{\partial|y|^{\nu}}d\nu+h(x,y,t), \tag{2}$$

under the following conditions:

$$\begin{aligned}&w(x,y,t)=0,\ (x,y,t)\in\partial\Theta\times(0,T],\\ &w(x,y,0)=0,\ w_{t}(x,y,0)=0,\ (x,y)\in\Theta,\end{aligned} \tag{3}$$

in which $\Theta=(0,L)\times(0,L)$ and $h(x,y,t)$ can be applied to show sources terms and sinks. Here, $Q_1(\gamma),Q_2(\beta),Q_3(\mu),Q_4(\nu)$ are the non-negative weight functions such that

$$\begin{aligned}&Q_{1}(\gamma)\ge 0,\ Q_{1}(\gamma)\not\equiv 0,\ 1<\gamma<2,\ \int_{1}^{2} Q_{1}(\gamma)d\gamma<\infty,\\ &Q_{2}(\beta)\ge 0,\ Q_{2}(\beta)\not\equiv 0,\ 1<\beta<2,\ \int_{1}^{2} Q_{2}(\beta)d\beta<\infty,\\ &Q_{3}(\mu)\ge 0,\ Q_{3}(\mu)\not\equiv 0,\ 0<\mu<1,\ \int_{0}^{1} Q_{3}(\mu)d\mu<\infty,\\ &Q_{4}(\nu)\ge 0,\ Q_{4}(\nu)\not\equiv 0,\ 0<\nu<1,\ \int_{0}^{1} Q_{4}(\nu)d\nu<\infty.\end{aligned} \tag{4}$$

Further, in Eq. (2), ${}^{C}\mathfrak{D}_{t}^{\alpha}$ stands for the Caputo fractional operator of order $1\le\alpha\le 2$, and the symbols $\frac{\partial^{\gamma}w(x,y,t)}{\partial|x|^{\gamma}}$ and $\frac{\partial^{\beta}w(x,y,t)}{\partial|y|^{\beta}}$ are the Riesz fractional operators over the $[0,L]$ by the following formulas:

$$\begin{aligned}\frac{\partial^{\gamma}w(x,y,t)}{\partial|x|^{\gamma}}&=C_{\gamma}\Big(\frac{1}{\Gamma(n-\gamma)}\frac{\partial^{n}}{\partial x^{n}}\int_{0}^{x}(x-\tau)^{n-\gamma-1}w(\tau,y,t)d\tau+\frac{(-1)^{n}}{\Gamma(n-\gamma)}\frac{\partial^{n}}{\partial x^{n}}\int_{x}^{L}(\tau-x)^{n-\gamma-1}w(\tau,y,t)d\tau\Big)\\ &=C_{\gamma}\Big({}_{0}\mathbb{D}_{x}^{\gamma}w+{}_{x}\mathbb{D}_{L}^{\gamma}w\Big),\\ \frac{\partial^{\beta}w(x,y,t)}{\partial|y|^{\beta}}&=C_{\beta}\Big(\frac{1}{\Gamma(n-\beta)}\frac{\partial^{n}}{\partial y^{n}}\int_{0}^{y}(y-\tau)^{n-\beta-1}w(\tau,y,t)d\tau+\frac{(-1)^{n}}{\Gamma(n-\beta)}\frac{\partial^{n}}{\partial y^{n}}\int_{y}^{L}(\tau-y)^{n-\gamma-1}w(\tau,y,t)d\tau\Big)\\ &=C_{\beta}\Big({}_{0}\mathbb{D}_{y}^{\gamma}w+{}_{y}\mathbb{D}_{L}^{\gamma}w\Big),\end{aligned} \tag{5}$$

where $C_{\gamma}=-\frac{1}{2\cos(\frac{\gamma\pi}{2})}$, $C_{\beta}=-\frac{1}{2\cos(\frac{\beta\pi}{2})}$, and ${}_{0}\mathbb{D}_{x}^{\gamma}$ and ${}_{x}\mathbb{D}_{L}^{\gamma}$ are the left and right Riemann-Liouville operators, respectively. To approximate the Riesz space fractional operator of distributed-order, we use the numerical formula based on the mid-point quadrature formula.

In this respect, we utilize the numerical formula based on the Gauss quadrature formula to estimate the Riesz space fractional operator of distributed-order. The Gauss quadrature formula is stable and has a higher computational efficiency and accuracy than the mid-point quadrature formula. To estimate the Riesz space fractional operator of distributed-order $\int_{\sigma_{\min}}^{\sigma_{\max}} Q(\sigma)\frac{\partial^{\sigma}w}{\partial|x|^{\sigma}}d\sigma$ we apply the Gauss quadrature formula, when the non-negative weight function $Q(\sigma)$ is a smooth function. When the non-negative weight function $Q(\sigma)$ is not a smooth function or has weak singularities, we can be rewritten the Riesz space fractional operator of distributed-order

$$\int_{\sigma_{\min}}^{\sigma_{\max}} Q(\sigma)\frac{\partial^{\sigma} w}{\partial|x|^{\sigma}}\varpi\varpi^{-1}d\sigma,$$

in which $\varpi=(\sigma-\sigma_{\min})^{a}(\sigma_{\max}-\sigma)^{b}$, $a,b>-1$ is the weight function such that the function $Q(\sigma)\frac{\partial^{\sigma} w}{\partial|x|^{\sigma}}\varpi$ is more smooth than $Q(\sigma)\frac{\partial^{\sigma} w}{\partial|x|^{\sigma}}$. The numerical formula based on Gauss quadrature formula for smooth and non-smooth solutions can obtain a high precision and accuracy. In this manuscript, we explain a difference-Legendre spectral method for the two-dimensional space-time distributed-order fractional diffusion-wave equations with the Riesz space fractional derivative. To get a high precision and efficient numerical technique to solve Eq. (2) with the conditions (3), the numerical formula based on the composite trapezoid rule is used to estimate the distribution-order integral part. Then, we apply an approximate formula which is displayed in [17] to approximate the fractional operator consisting of Caputo fractional derivative of order α. Thus, according to the above description, we can obtain the numerical approximation with high accuracy of second-order in time. Moreover, to estimate the distribution-order integral part consisting of Riesz space derivative, we apply the Gauss quadrature formula.

The outline of this manuscript is displayed as follows. In Section 2, we state the difference and Legendre spectral schemes for obtaining the numerical solution of Eq. (2). The stability and convergence analysis of the proposed method are demonstrated in Section 3. Some numerical examples to display the effectiveness of our numerical method are studied in Section 4. At the end, main conclusions are expressed in Section 5.

2. The difference and Legendre spectral schemes

In this section, we derive a numerical method based on the difference scheme and Legendre spectral scheme to solve Eq. (2). Let $\Delta t=\frac{T}{M}$ be the time step and $t_m=m\Delta t$ such that $m=0,1,\dots,M$, $M\in\mathbb{N}^{+}$. For $w(x,y,t)\in C\big(\Theta\times[0,T]\big)$ we consider $w^m=w^m(\cdot)=w(\cdot,t_m)$, and for simplicity and convenience we define the following notations

$$w^{m-\frac{1}{2}}=\frac{w^m+w^{m-1}}{2},\ \delta_t w^{m-\frac{1}{2}}=\frac{w^m-w^{m-1}}{\Delta t}. \tag{6}$$

We take into account Eq. (2) at $t=t_m$ and rewrite it as

$$\begin{aligned}\int_1^2 b(\alpha)^C\mathfrak{D}_t^{\alpha}w(x,y,t_m)d\alpha&=\int_1^2 Q_1(\gamma)\frac{\partial^{\gamma}w(x,y,t_m)}{\partial|x|^{\gamma}}d\gamma+\int_1^2 Q_2(\beta)\frac{\partial^{\beta}w(x,y,t_m)}{\partial|y|^{\beta}}d\beta\\&+\int_0^1 Q_3(\mu)\frac{\partial^{\mu}w(x,y,t_m)}{\partial|x|^{\mu}}d\mu+\int_0^1 Q_4(\nu)\frac{\partial^{\nu}w(x,y,t_m)}{\partial|y|^{\nu}}d\nu+h(x,y,t_m),\ m=0,1,\dots,M.\end{aligned}\tag{7}$$

In view of Eq. (6), we also consider an average of Eq. (7) at points $t=t_m$ and t_{m-1} to obtain

$$\int_1^2 b(\alpha)^C\mathfrak{D}_t^{\alpha}w^{m-\frac{1}{2}}d\alpha=\int_1^2 Q_1(\gamma)\frac{\partial^{\gamma}w^{m-\frac{1}{2}}}{\partial|x|^{\gamma}}d\gamma+\int_1^2 Q_2(\beta)\frac{\partial^{\beta}w^{m-\frac{1}{2}}}{\partial|y|^{\beta}}d\beta+\int_0^1 Q_3(\mu)\frac{\partial^{\mu}w^{m-\frac{1}{2}}}{\partial|x|^{\mu}}d\mu+\int_0^1 Q_4(\nu)\frac{\partial^{\nu}w^{m-\frac{1}{2}}}{\partial|y|^{\nu}}d\nu+h^{m-\frac{1}{2}},\ m=0,1,\dots,M. \tag{8}$$

To get an approximate solution for Eq. (2), we first use the composite trapezoid formula [15] to estimate the distributed-order integral part $\int_1^2 b(\alpha)^C\mathfrak{D}_t^{\alpha}w^{m-\frac{1}{2}}d\alpha$ in (8), and then we apply the Gauss quadrature to approximate the right-hand integrals in (8) to get

$$\begin{aligned}\Delta\alpha\sum_{k=0}^{2I}d_k b(\alpha_k)^C\mathfrak{D}_t^{\alpha_k}w^{m-\frac{1}{2}}+\mathcal{O}\big(\Delta\alpha^2\big)&=\frac{1}{2}\sum_{i=0}^{N_1}\tilde{\xi}_i^{(1)}Q_1(\gamma_i)\frac{\partial^{\gamma_i}w^{m-\frac{1}{2}}}{\partial|x|^{\gamma_i}}+\mathcal{O}\big(N_1^{-j_1}\big)\\&+\frac{1}{2}\sum_{i=0}^{N_2}\tilde{\xi}_i^{(2)}Q_2(\beta_i)\frac{\partial^{\beta_i}w^{m-\frac{1}{2}}}{\partial|y|^{\beta_i}}+\mathcal{O}\big(N_2^{-j_1}\big)\\&+\frac{1}{2}\sum_{i=0}^{N_3}\tilde{\xi}_i^{(3)}Q_3(\mu_i)\frac{\partial^{\mu_i}w^{m-\frac{1}{2}}}{\partial|x|^{\mu_i}}+\mathcal{O}\big(N_3^{-j_1}\big)\\&+\frac{1}{2}\sum_{i=0}^{N_4}\tilde{\xi}_i^{(4)}Q_4(\nu_i)\frac{\partial^{\nu_i}w^{m-\frac{1}{2}}}{\partial|y|^{\nu_i}}+\mathcal{O}\big(N_4^{-j_1}\big)+h^{m-\frac{1}{2}},\end{aligned}\tag{9}$$

where $b(\alpha)\in C^2([1,2])$, ${}^C\mathfrak{D}_t^{\alpha}w(x,t)|_{t=t_m}$ and ${}^C\mathfrak{D}_t^{\alpha}w(x,t)|_{t=t_{m-1}}\in C^2([1,2])$. Moreover, let

$$\begin{aligned}&Q_1(\gamma)=H_1^{\gamma-2}K_1\omega_1(\gamma),\ \omega_1(\gamma)=B_1\delta(\gamma-\gamma_1)+B_2(\gamma-\gamma_2),\ 1<\gamma_1<\gamma_2\le 2,\\&Q_2(\beta)=H_2^{\beta-2}K_2\omega_2(\beta),\ \omega_2(\beta)=B_3\delta(\beta-\beta_1)+B_4(\beta-\beta_2),\ 1<\beta_1<\beta_2\le 2,\\&Q_3(\mu)=H_3^{\mu-2}K_3\omega_3(\mu),\ \omega_3(\mu)=B_5\delta(\mu-\mu_1)+B_6(\mu-\mu_2),\ 0<\mu_1<\mu_2\le 1,\\&Q_4(\nu)=H_4^{\nu-2}K_4\omega_4(\nu),\ \omega_4(\nu)=B_7\delta(\nu-\nu_1)+B_8(\nu-\nu_2),\ 0<\nu_1<\nu_2\le 1,\end{aligned}\tag{10}$$

in which $[H_i]=cm$, $[K_i]=cm^2/s$ for $i=1,2,3,4$ are the dimensional positive constants and $B_j>0$, $j=1,2,3,4,5,6,7,8$. In the above discretization, $\Delta\alpha=\frac{1}{2I}$, $\alpha_k=1+k\Delta\alpha\,(0\le k\le 2I)$, $d_k=\frac{1}{2}\,(k=0,2I)$, $d_k=1\,(1\le k\le 2I-1)$, and γ_i, β_i, μ_i, ν_i are the Legendre-Gauss points such that $1\le\gamma_i,\beta_i\le 2$,

$0 \le \mu_i, \nu_i \le 1$. Also, $\tilde{\xi}_i^{(r)}, r = 1,2,3,4$ are the Legendre-Gauss weights given in [34] and $j_1 \le \min(N_r)+1, r = 1,2,3,4$. At this point, we set $\xi_i^{(1)} = \frac{\tilde{\xi}_i^{(1)} Q_1(\gamma_i)}{2}$, $\xi_i^{(2)} = \frac{\tilde{\xi}_i^{(2)} Q_2(\beta_i)}{2}$, $\xi_i^{(3)} = \frac{\tilde{\xi}_i^{(3)} Q_3(\mu_i)}{2}$, $\xi_i^{(4)} = \frac{\tilde{\xi}_i^{(4)} Q_4(\nu_i)}{2}$, and yield Eq. (9) in the following representation

$$\begin{aligned}\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) {}^C\mathfrak{D}_t^{\alpha_k} w^{m-\frac{1}{2}} + \mathcal{O}(\Delta\alpha^2) = &\sum_{i=0}^{N_1} \xi_i^{(1)} \frac{\partial^{\gamma_i} w^{m-\frac{1}{2}}}{\partial |x|^{\gamma_i}} + \mathcal{O}(N_1^{-j_1}) \\ &+ \sum_{i=0}^{N_2} \xi_i^{(2)} \frac{\partial^{\beta_i} w^{m-\frac{1}{2}}}{\partial |y|^{\beta_i}} + \mathcal{O}(N_2^{-j_1}) \\ &+ \sum_{i=0}^{N_3} \xi_i^{(3)} \frac{\partial^{\mu_i} w^{m-\frac{1}{2}}}{\partial |x|^{\mu_i}} + \mathcal{O}(N_3^{-j_1}) \\ &+ \sum_{i=0}^{N_4} \xi_i^{(4)} \frac{\partial^{\nu_i} w^{m-\frac{1}{2}}}{\partial |y|^{\nu_i}} + \mathcal{O}(N_4^{-j_1}) + h^{m-\frac{1}{2}}.\end{aligned} \tag{11}$$

Here, by using the fully discrete difference method [17, Eq. (2.7)], we approximate the term $\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) {}^C\mathfrak{D}_t^{\alpha_k} w^{m-\frac{1}{2}}$ as

$$\begin{aligned}\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{m-1} \vartheta_l^{(\sigma_k)} \delta_t w^{m-l-\frac{1}{2}} + \mathcal{O}((\Delta t)^2) + \mathcal{O}(\Delta\alpha^2) = &\sum_{i=0}^{N_1} \xi_i^{(1)} \frac{\partial^{\gamma_i} w^{m-\frac{1}{2}}}{\partial |x|^{\gamma_i}} + \mathcal{O}(N_1^{-j_1}) \\ &+ \sum_{i=0}^{N_2} \xi_i^{(2)} \frac{\partial^{\beta_i} w^{m-\frac{1}{2}}}{\partial |y|^{\beta_i}} + \mathcal{O}(N_2^{-j_1}) \\ &+ \sum_{i=0}^{N_3} \xi_i^{(3)} \frac{\partial^{\mu_i} w^{m-\frac{1}{2}}}{\partial |x|^{\mu_i}} + \mathcal{O}(N_3^{-j_1}) \\ &+ \sum_{i=0}^{N_4} \xi_i^{(4)} \frac{\partial^{\nu_i} w^{m-\frac{1}{2}}}{\partial |y|^{\nu_i}} + \mathcal{O}(N_4^{-j_1}) + h^{m-\frac{1}{2}},\end{aligned} \tag{12}$$

where

$$\vartheta_0^{(\sigma_k)} = \left(1 + \frac{\sigma_k}{2}\right) f_0^{(\sigma_k)},\ \vartheta_l^{(\sigma_k)} = \left(1 + \frac{\sigma_k}{2}\right) f_l^{(\sigma_k)} - \frac{\sigma_k}{2} f_{l-1}^{(\sigma_k)}, \quad l \ge 1, \quad \sigma_k = \alpha_k - 1 \ \ 0 \le k \le 2I, \tag{13}$$

and

$$f_0^{(\sigma_k)} = 1,\ f_l^{(\sigma_k)} = \left(1 - \frac{\sigma_k + 1}{l}\right) f_{l-1}^{(\sigma_k)},\ l \ge 1. \tag{14}$$

In order to know and study the behaviors of (12), we define the fractional Sobolev space $\mathfrak{H}^{\varsigma}(\mathbb{R}^2)$, $\varsigma \ge 0$ on $\mathbb{R}^2$

$$\mathfrak{H}^{\varsigma}(\mathbb{R}^2) = \{w \in L^2(\mathbb{R}^2) | (1+|s|^2)^{\frac{\varsigma}{2}} \mathcal{F}(w)(s) \in L^2(\mathbb{R}^2)\},$$

subject to the following norm

$$\| w \|_{\mathfrak{H}^{\varsigma}(\mathbb{R}^2)} = \| (1+|s|^2)^{\frac{\varsigma}{2}} \mathcal{F}(w)(s) \|_{L^2(\mathbb{R}^2)},$$

where $\mathcal{F}$ is the Fourier transform of w. Also, the fractional Sobolev space $\mathfrak{H}^{\varsigma}(\Theta)$, $\varsigma \ge 0$ on Θ is shown by

$$\mathfrak{H}^{\varsigma}(\Theta) = \{w \in L^2(\Theta) | \exists \tilde{w} \in \mathfrak{H}^{\varsigma}(\mathbb{R}^2)\, s.t.\, \tilde{w}|_{\Theta} = w\},$$

with the norm

$$\| w \|_{\mathfrak{H}^{\varsigma}(\Theta)} = \inf_{\tilde{w} \in \mathfrak{H}^{\varsigma}(\mathbb{R}^2)\ \tilde{w}|_{\Theta} = w} \| \tilde{w} \|_{\mathfrak{H}^{\varsigma}(\mathbb{R}^2)},$$

where $\mathfrak{H}_0^{\varsigma}(\Theta)$ illustrates the closure of $\mathbb{C}_0^{\infty}(\Theta)$ subject to the norm $\| w \|_{\mathfrak{H}^{\varsigma}(\Theta)}$. We now suppose that

$$\begin{aligned}\mathbf{F}_x w^{m-\frac{1}{2}} &= \sum_{i=0}^{N_1} \xi_i^{(1)} \frac{\partial^{\gamma_i} w^{m-\frac{1}{2}}}{\partial |x|^{\gamma_i}} + \sum_{i=0}^{N_3} \xi_i^{(3)} \frac{\partial^{\mu_i} w^{m-\frac{1}{2}}}{\partial |x|^{\mu_i}}, \\ \mathbf{F}_y w^{m-\frac{1}{2}} &= \sum_{i=0}^{N_2} \xi_i^{(2)} \frac{\partial^{\beta_i} w^{m-\frac{1}{2}}}{\partial |y|^{\beta_i}} + \sum_{i=0}^{N_4} \xi_i^{(4)} \frac{\partial^{\nu_i} w^{m-\frac{1}{2}}}{\partial |y|^{\nu_i}},\end{aligned} \tag{15}$$

and change Eq. (12) to the following equation

$$\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{m-1} \vartheta_l^{(\sigma_k)} \delta_t w^{m-l-\frac{1}{2}} = \left(\mathbf{F}_x + \mathbf{F}_y\right) w^{m-\frac{1}{2}} + h^{m-\frac{1}{2}} + \mathcal{O}\left((\Delta t)^2 + (\Delta\alpha)^2 + (\min(N_r))^{-j_1}\right), \tag{16}$$

which by considering Eq. (6) gives rise to

$$\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{m-1} \vartheta_l^{(\sigma_k)} \left[\frac{w^{m-l} - w^{m-l-1}}{\Delta t}\right] = \left(\mathbf{F}_x + \mathbf{F}_y\right) \left[\frac{w^m + w^{m-1}}{2}\right] + h^{m-\frac{1}{2}} + \mathcal{O}\left((\Delta t)^2 + (\Delta\alpha)^2 + (\min(N_r))^{-j_1}\right), \tag{17}$$

or in other words

$$\Delta\alpha\sum_{k=0}^{2I} d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\sum_{l=0}^{m-1}\vartheta_l^{(\sigma_k)}\Big[w^{m-l}-w^{m-l-1}\Big]=\Delta t\Big(\mathbf{F}_x+\mathbf{F}_y\Big)\Big[\frac{w^m+w^{m-1}}{2}\Big]+\Delta t h^{m-\frac{1}{2}}+\mathcal{O}\big((\Delta t)^3+\Delta t(\Delta\alpha)^2+\Delta t(\min(N_r))^{-j_1}\big). \tag{18}$$

With the intention of obtaining an equivalent form for the above relation, we set

$$\zeta_l=\Delta\alpha\sum_{k=0}^{2I} d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\vartheta_l^{(\sigma_k)},$$

and add the left-hand side of Eq. (18) with the perturbation term $\Big(\frac{(\Delta t)^3\mathbf{F}_x\mathbf{F}_y}{4}-\frac{\zeta_0(\Delta t)^2}{2}\mathbf{F}_y\Big)\delta_t w^{m-\frac{1}{2}}$, to get

$$\Big(\zeta_0-\frac{\Delta t}{2}\mathbf{F}_x\Big)\Big(1-\frac{\Delta t}{2}\mathbf{F}_y\Big)w^m=\Big[\Big(1+\frac{\Delta t}{2}\mathbf{F}_x\Big)\Big(1+\frac{\Delta t}{2}\mathbf{F}_y\Big)+\zeta_0-\zeta_1-\frac{\zeta_0\Delta t}{2}\mathbf{F}_y-1\Big]w^{m-1}+\sum_{l=2}^{m-1}(\zeta_{l-1}-\zeta_l)w^{m-l}$$
$$+\Delta t\, h^{m-\frac{1}{2}}+\mathcal{O}\big((\Delta t)^3+\Delta t(\Delta\alpha)^2+\Delta t(\min(N_r))^{-j_1}\big). \tag{19}$$

At this stage, we recall the following basis function for the spatial discretization in x and y directions [43]

$$\Phi_j(x)=L_j(\hat{x})-L_{j+2}(\hat{x}),\ 0\le\hat{x}\le L,\ \hat{x}=\frac{2x-L}{L}\in[-1,1],$$
$$\Phi'_j(y)=L_j(\hat{y})-L_{j+2}(\hat{y}),\ 0\le\hat{y}\le L,\ \hat{y}=\frac{2y-L}{L}\in[-1,1], \tag{20}$$

where $j=0,1,\dots,N-2$ and L_j is the Legendre basis function [34]. We construct the function space

$$\mathbb{S}_N=\mathrm{span}\{\Phi_j(x)\Phi_{j'}(y),\ j,j'=0,1,\dots,N-2\}$$

and ignore the error part in Eq. (19). In this sense, we can get the fully discrete method for Eq. (2) to find w_N^m, $m=0,1,\dots,M-1$, as follows

$$\Big(\Big(\zeta_0-\frac{\Delta t}{2}\mathbf{F}_x\Big)\Big(1-\frac{\Delta t}{2}\mathbf{F}_y\Big)w_N^m,\varsigma\Big)=\Big(\Big[\Big(1+\frac{\Delta t}{2}\mathbf{F}_x\Big)\Big(1+\frac{\Delta t}{2}\mathbf{F}_y\Big)+\zeta_0-\zeta_1-\frac{\zeta_0\Delta t}{2}\mathbf{F}_y-1\Big]w_N^{m-1},\varsigma\Big)$$
$$+\sum_{l=2}^{m-1}(\zeta_{l-1}-\zeta_l)\Big(w_N^{m-l},\varsigma\Big)+\Delta t\Big(h^{m-\frac{1}{2}},\varsigma\Big),\ \forall\varsigma\in\mathbb{S}_N, \tag{21}$$
$$w_N^0=\Lambda_N^{1,0}w_0,$$

where $\Lambda_N^{1,0}$ is the orthogonal projection operator from $\mathfrak{H}_0^1$ onto $\mathbb{S}_N^{1,0}$ for the scalar product associated with the norm $|\cdot|_{\mathfrak{H}^1}$ and satisfy the following relation

$$(\Lambda_N^{1,0}w-w,\varsigma)+\big(\partial_x(\Lambda_N^{1,0}w-w),\partial_x\varsigma\big)=0,\ \varsigma\in\mathbb{S}_N^{1,0}. \tag{22}$$

$\mathbb{S}_N^{1,0}$ stands for the space $\mathbb{S}_N\bigcap\mathfrak{H}_0^1$, i.e., for the space of polynomials in $\mathbb{S}_N$ which vanish in ±1 together with their derivatives. The space $\mathbb{S}_N^{1,0}$ is also denoted by $\mathbb{S}_N$ for the sake of simplicity. Finally, we estimate the approximated solution w_N^m as

$$w_N^m=\sum_{r_1=0}^{N-2}\sum_{r_2=0}^{N-2}a_{r_1,r_2}^m\Phi_{r_1}(x)\Phi'_{r_2}(y), \tag{23}$$

and consider $\varsigma=\Phi_{r_1}(x)\Phi'_{r_2}(y)\,(r_1,r_2=0,1,\dots,N-2)$ to derive the matrix representation as

$$\Big(\zeta_0\mathbf{M}_x-\frac{\Delta t}{2}\mathbf{Q}_x\Big)\mathbf{A}^m\Big(\mathbf{M}_y-\frac{\Delta t}{2}\mathbf{Q}_y\Big)^T=\Big[\Big(\mathbf{M}_x+\frac{\Delta t}{2}\mathbf{Q}_x\Big)\Big(\mathbf{M}_y+\frac{\Delta t}{2}\mathbf{Q}_y\Big)^T+\Big(\zeta_0-\zeta_1-1\Big)\mathbf{M}_y-\frac{\zeta_0\Delta t}{2}\mathbf{Q}_y\Big]\mathbf{A}^{m-1}+\sum_{l=2}^{m-1}\mathbf{A}^{m-l}+\Delta t\mathbf{H}^m, \tag{24}$$

where $\mathbf{M}_x,\mathbf{Q}_x,\mathbf{A}^m,\mathbf{H}^m\in\mathbb{R}^{(N-1)\times(N-1)}$ are given by

$$\Big(\mathbf{M}_x\Big)_{r_1,r_2}=\big(\Phi_{r_1}(x),\Phi'_{r_2}(y)\big),\ \Big(\mathbf{M}_y\Big)_{r_1,r_2}=\big(\Phi_{r_1}(x),\Phi'_{r_2}(y)\big),$$
$$\Big(\mathbf{Q}_x\Big)_{r_1,r_2}=\kappa_1\sum_{i=0}^{N_1}\xi_i^{(1)}C_{\gamma_i}\Big(\mathbf{Q}_x^{(\gamma_i)}+\big(\mathbf{Q}_x^{(\gamma_i)}\big)^T\Big)+\kappa_3\sum_{i=0}^{N_3}\xi_i^{(3)}C_{\mu_i}\Big(\mathbf{Q}_x^{(\mu_i)}+\big(\mathbf{Q}_x^{(\mu_i)}\big)^T\Big),$$
$$\Big(\mathbf{Q}_y\Big)_{r_1,r_2}=\kappa_2\sum_{i=0}^{N_2}\xi_i^{(2)}C_{\beta_i}\Big(\mathbf{Q}_y^{(\beta_i)}+\big(\mathbf{Q}_y^{(\beta_i)}\big)^T\Big)+\kappa_4\sum_{i=0}^{N_4}\xi_i^{(4)}C_{\nu_i}\Big(\mathbf{Q}_y^{(\nu_i)}+\big(\mathbf{Q}_y^{(\nu_i)}\big)^T\Big), \tag{25}$$
$$\Big(\mathbf{A}^m\Big)_{r_1,r_2}=a_{r_1,r_2}^m,\ \Big(\mathbf{H}^m\Big)_{r_1,r_2}=\Big(h^{m-\frac{1}{2}},\Phi_{r_1}\Phi'_{r_2}\Big),$$

and

$$\big(\mathbf{Q}_x^{(\gamma_i)}\big)_{r_1,r_2}=\big({}_0\mathbb{D}_x^{\gamma_i}\Phi_{r_1},{}_x\mathbb{D}_L^{\gamma_i}\Phi'_{r_2}\big),\ \big(\mathbf{Q}_x^{(\mu_i)}\big)_{r_1,r_2}=\big({}_0\mathbb{D}_x^{\mu_i}\Phi_{r_1},{}_x\mathbb{D}_L^{\mu_i}\Phi'_{r_2}\big),$$
$$\big(\mathbf{Q}_y^{(\beta_i)}\big)_{r_1,r_2}=\big({}_0\mathbb{D}_y^{\beta_i}\Phi_{r_1},{}_y\mathbb{D}_L^{\beta_i}\Phi'_{r_2}\big),\ \big(\mathbf{Q}_y^{(\nu_i)}\big)_{r_1,r_2}=\big({}_0\mathbb{D}_y^{\nu_i}\Phi_{r_1},{}_y\mathbb{D}_L^{\nu_i}\Phi'_{r_2}\big). \tag{26}$$

We should mention that, by using the Legendre polynomial properties and the orthogonal conditions for this polynomial, we can easily compute the matrices $\mathbf{M}_x$ and $\mathbf{M}_y$. Moreover, by applying the Jacobi-Gauss quadrature, we can compute the matrices $\mathbf{Q}_x^{(\gamma_i)}$, $\mathbf{Q}_x^{(\mu_i)}$, $\mathbf{Q}_y^{(\beta_i)}$ and $\mathbf{Q}_y^{(\nu_i)}$. Also, by calculating the matrices $\mathbf{Q}_x^{(\gamma_i)}$, $\mathbf{Q}_x^{(\mu_i)}$, $\mathbf{Q}_y^{(\beta_i)}$ and $\mathbf{Q}_y^{(\nu_i)}$, the matrices $\mathbf{Q}_x$ and $\mathbf{Q}_y$ are consequently obtained.

3. Stability and convergence analyses

This section shows the stability and convergence analyses of the proposed method for Eq. (2). At first, we consider

$$\begin{aligned}
&\mathcal{H}_x^{\gamma_i}(w,\varsigma)=\left({}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}w,{}_x\mathbb{D}_L^{\frac{\gamma_i}{2}}\varsigma\right)+\left({}_x\mathbb{D}_L^{\frac{\gamma_i}{2}}w,{}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}\varsigma\right),\\
&\mathcal{H}_y^{\beta_i}(w,\varsigma)=\left({}_0\mathbb{D}_y^{\frac{\beta_i}{2}}w,{}_y\mathbb{D}_L^{\frac{\beta_i}{2}}\varsigma\right)+\left({}_y\mathbb{D}_L^{\frac{\beta_i}{2}}w,{}_0\mathbb{D}_y^{\frac{\beta_i}{2}}\varsigma\right),\\
&\mathcal{H}_x^{\mu_i}(w,\varsigma)=\left({}_0\mathbb{D}_x^{\frac{\mu_i}{2}}w,{}_x\mathbb{D}_L^{\frac{\mu_i}{2}}\varsigma\right)+\left({}_x\mathbb{D}_L^{\frac{\mu_i}{2}}w,{}_0\mathbb{D}_x^{\frac{\mu_i}{2}}\varsigma\right),\\
&\mathcal{H}_y^{\nu_i}(w,\varsigma)=\left({}_0\mathbb{D}_y^{\frac{\nu_i}{2}}w,{}_y\mathbb{D}_L^{\frac{\nu_i}{2}}\varsigma\right)+\left({}_y\mathbb{D}_L^{\frac{\nu_i}{2}}w,{}_0\mathbb{D}_y^{\frac{\nu_i}{2}}\varsigma\right),
\end{aligned}\tag{27}$$

and assume that

$$\mathcal{H}(w,\varsigma)=\kappa_1\sum_{i=0}^{N_1}\xi_i^{(1)}C_{\gamma_i}\mathcal{H}_x^{\gamma_i}(w,\varsigma)+\kappa_2\sum_{i=0}^{N_2}\xi_i^{(2)}C_{\beta_i}\mathcal{H}_y^{\beta_i}(w,\varsigma)+\kappa_3\sum_{i=0}^{N_3}\xi_i^{(3)}C_{\mu_i}\mathcal{H}_x^{\mu_i}(w,\varsigma)+\kappa_4\sum_{i=0}^{N_4}\xi_i^{(4)}C_{\nu_i}\mathcal{H}_y^{\nu_i}(w,\varsigma).\tag{28}$$

We define the orthogonal projection operator $\Lambda_N^{P,Q}:\mathfrak{H}_0^{\frac{\gamma_{N_1}}{2}}(\Theta)\bigcap\mathfrak{H}_0^{\frac{\beta_{N_2}}{2}}(\Theta)\rightarrow\mathbb{S}_N$ by

$$\mathcal{H}(w-\Lambda_N^{P,Q}w,\varsigma)=0,\ \forall\varsigma\in\mathbb{S}_N,\tag{29}$$

and consider

$$\begin{aligned}
&\left|w\right|_{P,Q}=\left[\kappa_1\sum_{i=0}^{N_1}\xi_i^{(1)}\parallel{}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}w\parallel_{L^2(\Theta)}+\kappa_2\sum_{i=0}^{N_2}\xi_i^{(2)}\parallel{}_0\mathbb{D}_y^{\frac{\beta_i}{2}}w\parallel_{L^2(\Theta)}+\kappa_3\sum_{i=0}^{N_3}\xi_i^{(3)}\parallel{}_0\mathbb{D}_x^{\frac{\mu_i}{2}}w\parallel_{L^2(\Theta)}+\kappa_4\sum_{i=0}^{N_4}\xi_i^{(4)}\parallel{}_0\mathbb{D}_y^{\frac{\nu_i}{2}}w\parallel_{L^2(\Theta)}\right]^{\frac{1}{2}},\\
&\parallel w\parallel_{P,Q}=\left(\parallel w\parallel_{L^2(\Theta)}^2+\left|w\right|_{P,Q}^2\right)^{\frac{1}{2}}.
\end{aligned}\tag{30}$$

In the sequel, we state some lemmas which are considered as the main preliminaries to express the stability and convergence theorems.

Lemma 3.1. *[43] Assume that $\tau\in(0,1]$, if $w\in\mathfrak{H}_0^\tau(\Theta)$, there exist positive constants $\mathbb{K}_1<1$ and $\mathbb{K}_2$ independent of w, such that*

$$\mathbb{K}_1\parallel w\parallel_{\mathfrak{H}_0^\tau(\Theta)}\le\left|w\right|_{\mathfrak{H}_0^\tau(\Theta)}\le\parallel w\parallel_{\mathfrak{H}_0^\tau(\Theta)}\le\mathbb{K}_2|w|_{\mathfrak{H}_0^{\max(\tau)}(\Theta)}.\tag{31}$$

Lemma 3.2. *[4] Suppose that $0\le s'\le q,\ s',q\in\mathbb{R}$. Then, for each $w\in\mathfrak{H}_0^{s'}(\Theta)\bigcap\mathfrak{H}^q(\Theta)$, we have*

$$\parallel w-\Lambda_N^{1,0}w\parallel_{s'}\le\mathbb{K}N^{s'-q}\parallel w\parallel_q,\tag{32}$$

where $\mathbb{K}>0$ is dependent on q.

Lemma 3.3. *Suppose that $0<\frac{\gamma_i}{2},\frac{\beta_{i'}}{2}<1<q$, such that $\frac{\gamma_i}{2},\frac{\beta_{i'}}{2}\neq\frac{1}{2}$ for $i=0,1,\dots,N_1$ and $i'=0,1,\dots,N_2$. Then for each $w\in\mathfrak{H}_0^{\frac{\gamma_{N_1}}{2}}(\Theta)\bigcap\mathfrak{H}_0^{\frac{\beta_{N_2}}{2}}(\Theta)\bigcap\mathfrak{H}^q(\Theta)$, we have*

$$\left|w-\Lambda_N^{P,Q}w\right|_{P,Q}\le\mathbb{K}\Big[\sum_{i=0}^{N_1}\xi_i^{(1)}N^{\frac{\gamma_i}{2}-q}\parallel w\parallel_q+\sum_{i=0}^{N_2}\xi_i^{(2)}N^{\frac{\beta_i}{2}-q}\parallel w\parallel_q+\sum_{i=0}^{N_3}\xi_i^{(3)}N^{\frac{\mu_i}{2}-q}\parallel w\parallel_q+\sum_{i=0}^{N_4}\xi_i^{(4)}N^{\frac{\nu_i}{2}-q}\parallel w\parallel_q\Big],\tag{33}$$

where $\mathbb{K}>0$ is independent of N.

Proof. According to the result studied in [43], we have

$$\begin{aligned}
\left|w-\Lambda_N^{P,Q}w\right|_{P,Q}^2&=\mathcal{H}(w-\Lambda_N^{P,Q}w,w-\Lambda_N^{P,Q}w)=\mathcal{H}(w-\Lambda_N^{P,Q}w,w-w_N)\\
&\le\mathbb{K}\left|w-\Lambda_N^{P,Q}w\right|_{P,Q}\left|w-w_N\right|_{P,Q}.
\end{aligned}\tag{34}$$

We now consider $w_N=\Lambda_N^{1,0}w$ and apply Lemmas 3.1 and 3.2 to obtain

$$\begin{aligned}
\left|w-\Lambda_N^{P,Q}w\right|_{P,Q}&\le\mathbb{K}\left|w-\Lambda_N^{1,0}w\right|_{P,Q}\\
&\le\mathbb{K}\Big[\sum_{i=0}^{N_1}\xi_i^{(1)}N^{\frac{\gamma_i}{2}-q}\parallel w\parallel_q+\sum_{i=0}^{N_2}\xi_i^{(2)}N^{\frac{\beta_i}{2}-q}\parallel w\parallel_q+\Big[\sum_{i=0}^{N_3}\xi_i^{(3)}N^{\frac{\mu_i}{2}-q}\parallel w\parallel_q+\sum_{i=0}^{N_4}\xi_i^{(4)}N^{\frac{\nu_i}{2}-q}\parallel w\parallel_q\Big].
\end{aligned}\tag{35}$$

Here the proof of this lemma is completed. □

Lemma 3.4. *[12] For each $w \in \mathfrak{H}_0^{\tau}(\Theta)$ and $w_1 \in \mathfrak{H}_0^{\frac{\tau}{2}}(\Theta)$, we have*

$$\left({}_0\mathbb{D}_x^{\tau} w, w_1\right) = \left({}_0\mathbb{D}_x^{\frac{\tau}{2}} w, {}_x\mathbb{D}_L^{\frac{\tau}{2}} w_1\right),\ \left({}_x\mathbb{D}_L^{\tau} w, w_1\right) = \left({}_x\mathbb{D}_L^{\frac{\tau}{2}} w, {}_0\mathbb{D}_x^{\frac{\tau}{2}} w_1\right). \tag{36}$$

Lemma 3.5. *[37] Let $\tau \geq 0$ and $w \in C_0^{\infty}(\Theta)$. Then*

$$\begin{aligned} &\left({}_0\mathbb{D}_x^{\tau} w, {}_x\mathbb{D}_L^{\tau} w\right) = \cos(\tau\pi) \parallel {}_{-\infty}\mathbb{D}_x^{\tau} \hat{w} \parallel_{L^2(\Theta)}^2 = \cos(\tau\pi) \parallel {}_x\mathbb{D}_{\infty}^{\tau} \hat{w} \parallel_{L^2(\Theta)}^2, \\ &\left({}_0\mathbb{D}_y^{\tau} w, {}_y\mathbb{D}_L^{\tau} w\right) = \cos(\tau\pi) \parallel {}_{-\infty}\mathbb{D}_y^{\tau} \hat{w} \parallel_{L^2(\Theta)}^2 = \cos(\tau\pi) \parallel {}_y\mathbb{D}_{\infty}^{\tau} \hat{w} \parallel_{L^2(\Theta)}^2, \end{aligned} \tag{37}$$

in which $\hat{w}$ is a extension of w by zero outside Θ.

Lemma 3.6. *[43] Under the assumptions of Lemma 3.5, we have*

$$\begin{aligned} &\left({}_0\mathbb{D}_x^{\tau_1}\, {}_0\mathbb{D}_y^{\tau_2} w, {}_x\mathbb{D}_L^{\tau_1}\, {}_y\mathbb{D}_L^{\tau_2} w\right) = \cos(\tau_1\pi)\cos(\tau_2\pi) \parallel {}_{-\infty}\mathbb{D}_x^{\tau_1}\, {}_{-\infty}\mathbb{D}_y^{\tau_2} \hat{w} \parallel_{L^2(\Theta)}^2, \\ &\left({}_0\mathbb{D}_x^{\tau_1}\, {}_y\mathbb{D}_L^{\tau_2} w, {}_x\mathbb{D}_L^{\tau_1}\, {}_0\mathbb{D}_y^{\tau_2} w\right) = \cos(\tau_1\pi)\cos(\tau_2\pi) \parallel {}_{-\infty}\mathbb{D}_x^{\tau_1}\, {}_{-\infty}\mathbb{D}_y^{\tau_2} \hat{w} \parallel_{L^2(\Theta)}^2, \end{aligned} \tag{38}$$

in which $\hat{w}$ is a extension of w by zero outside Θ.

Theorem 3.7. *Suppose that w^m be the solution of Eq. (19). Then, there exists $\mathbb{K} > 0$, independent of m and Δt, such that*

$$\parallel w^m \parallel_{P,Q} \leq \mathbb{K}\left[\left|w^0\right|_{P,Q} + \Delta t \sum_{j=1}^{m} \parallel h^{j-\frac{1}{2}} \parallel\right]. \tag{39}$$

Proof. First, we take into account Eq. (16) in the semi-discrete form and show

$$\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{j-1} \vartheta_l^{(\sigma_k)} \left(\delta_t w^{j-l-\frac{1}{2}}, \varsigma\right) + \frac{(\Delta t)^2}{4}\left(\mathbf{F}_x\mathbf{F}_y \delta_t w^{j-\frac{1}{2}}, \varsigma\right) - \frac{\zeta_0 \Delta t}{2}\left(\mathbf{F}_y \delta_t w^{j-\frac{1}{2}}, \varsigma\right) = \left(\left(\mathbf{F}_x + \mathbf{F}_y\right) w^{j-\frac{1}{2}}, \varsigma\right) + \left(h^{j-\frac{1}{2}}, \varsigma\right), \tag{40}$$

which by setting $\varsigma = \delta_t w^{j-\frac{1}{2}}$ it is represented as

$$\begin{aligned} &\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{j-1} \vartheta_l^{(\sigma_k)} \left(\delta_t w^{j-l-\frac{1}{2}}, \delta_t w^{j-\frac{1}{2}}\right) + \frac{(\Delta t)^2}{4}\left(\mathbf{F}_x\mathbf{F}_y \delta_t w^{j-\frac{1}{2}}, \delta_t w^{j-\frac{1}{2}}\right) - \frac{\zeta_0 \Delta t}{2}\left(\mathbf{F}_y \delta_t w^{j-\frac{1}{2}}, \delta_t w^{j-\frac{1}{2}}\right) \\ &\qquad = \left(\left(\mathbf{F}_x + \mathbf{F}_y\right) w^{j-\frac{1}{2}}, \delta_t w^{j-\frac{1}{2}}\right) + \left(h^{j-\frac{1}{2}}, \delta_t w^{j-\frac{1}{2}}\right). \end{aligned} \tag{41}$$

Next, we intend to get the value $\left(\mathbf{F}_x\mathbf{F}_y\varsigma, \varsigma\right) \geq 0$ is non-negative. For this purpose, we apply Lemma 3.4 and obtain

$$\begin{aligned} \left(\mathbf{F}_x\mathbf{F}_y\varsigma, \varsigma\right) &= 2\sum_{i=0}^{N_1}\sum_{i'=0}^{N_2} \xi_i^{(1)} C_{\gamma_i} \xi_{i'}^{(2)} C_{\beta_{i'}} \mathcal{H}^{(\gamma_i,\beta_{i'})}(\varsigma,\varsigma) \\ &+ 2\sum_{i=0}^{N_1}\sum_{i'=0}^{N_4} \xi_i^{(1)} C_{\gamma_i} \xi_{i'}^{(4)} C_{\nu_{i'}} \mathcal{H}^{(\gamma_i,\nu_{i'})}(\varsigma,\varsigma) \\ &+ 2\sum_{i=0}^{N_2}\sum_{i'=0}^{N_3} \xi_i^{(2)} C_{\beta_i} \xi_{i'}^{(3)} C_{\mu_{i'}} \mathcal{H}^{(\beta_i,\mu_{i'})}(\varsigma,\varsigma) \\ &+ 2\sum_{i=0}^{N_3}\sum_{i'=0}^{N_4} \xi_i^{(3)} C_{\mu_i} \xi_{i'}^{(4)} C_{\nu_{i'}} \mathcal{H}^{(\mu_i,\nu_{i'})}(\varsigma,\varsigma), \end{aligned} \tag{42}$$

where

$$\mathcal{H}^{(\gamma_i,\beta_{i'})}(\varsigma,\varsigma) = \left({}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_y\mathbb{D}_L^{\frac{\beta_{i'}}{2}} \varsigma, {}_x\mathbb{D}_L^{\frac{\gamma_i}{2}}\, {}_0\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \varsigma\right) + \left({}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_0\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \varsigma, {}_x\mathbb{D}_L^{\frac{\gamma_i}{2}}\, {}_y\mathbb{D}_L^{\frac{\beta_{i'}}{2}} \varsigma\right). \tag{43}$$

Also, from Lemma 3.6 we have

$$\begin{aligned} 2C_{\gamma_i} C_{\beta_{i'}} \mathcal{H}^{(\gamma_i,\beta_{i'})}(\varsigma,\varsigma) &= 2C_{\gamma_i} C_{\beta_{i'}} \left[\left({}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_y\mathbb{D}_L^{\frac{\beta_{i'}}{2}} \varsigma, {}_x\mathbb{D}_L^{\frac{\gamma_i}{2}}\, {}_0\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \varsigma\right) + \left({}_0\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_0\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \varsigma, {}_x\mathbb{D}_L^{\frac{\gamma_i}{2}}\, {}_y\mathbb{D}_L^{\frac{\beta_{i'}}{2}} \varsigma\right)\right] \\ &= 4C_{\gamma_i} C_{\beta_{i'}} \cos\left(\frac{\gamma_i\pi}{2}\right)\cos\left(\frac{\beta_{i'}\pi}{2}\right) \parallel {}_{-\infty}\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_{-\infty}\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \hat{\varsigma} \parallel_{L^2(\Theta)}^2 \\ &= \parallel {}_{-\infty}\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_{-\infty}\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \hat{\varsigma} \parallel_{L^2(\Theta)}^2, \end{aligned} \tag{44}$$

which implies that

$$\left(\mathbf{F}_x\mathbf{F}_y\varsigma, \varsigma\right) = 2\sum_{i=0}^{N_1}\sum_{i'=0}^{N_2} \xi_i^{(1)} \xi_{i'}^{(2)} \parallel {}_{-\infty}\mathbb{D}_x^{\frac{\gamma_i}{2}}\, {}_{-\infty}\mathbb{D}_y^{\frac{\beta_{i'}}{2}} \hat{\varsigma} \parallel_{L^2(\Theta)}^2$$

$$+2\sum_{i=0}^{N_1}\sum_{i'=0}^{N_4}\xi_i^{(1)}\xi_{i'}^{(4)}\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\gamma_i}{2}}\,{}_{-\infty}\mathbb{D}_y^{\frac{\nu_{i'}}{2}}\,\hat{\varsigma}\parallel_{L^2(\Theta)}^2$$
$$+2\sum_{i=0}^{N_2}\sum_{i'=0}^{N_3}\xi_i^{(2)}\xi_{i'}^{(3)}\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\beta_i}{2}}\,{}_{-\infty}\mathbb{D}_y^{\frac{\mu_{i'}}{2}}\,\hat{\varsigma}\parallel_{L^2(\Theta)}^2$$
$$+2\sum_{i=0}^{N_3}\sum_{i'=0}^{N_4}\xi_i^{(3)}\xi_{i'}^{(4)}\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\mu_i}{2}}\,{}_{-\infty}\mathbb{D}_y^{\frac{\nu_{i'}}{2}}\,\hat{\varsigma}\parallel_{L^2(\Theta)}^2\geq 0. \tag{45}$$

Moreover, from Lemmas 3.4 and 3.5, we calculate the following value

$$\begin{aligned}\Big(\mathbf{F}_x w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big) &= \Big(\sum_{i=0}^{N_1}\xi_i^{(1)}\frac{\partial^{\gamma_i}w^{j-\frac{1}{2}}}{\partial|x|^{\gamma_i}},\delta_t w^{j-\frac{1}{2}}\Big)+\Big(\sum_{i=0}^{N_3}\xi_i^{(3)}\frac{\partial^{\mu_i}w^{j-\frac{1}{2}}}{\partial|x|^{\mu_i}},\delta_t w^{j-\frac{1}{2}}\Big)\\
&=\frac{1}{2\Delta t}\sum_{i=0}^{N_1}\xi_i^{(1)}C_{\gamma_i}\big(\mathcal{H}_x^{(\gamma_i)}(w^j,w^j)-\mathcal{H}_x^{(\gamma_i)}(w^{j-1},w^{j-1})\big)\\
&+\frac{1}{2\Delta t}\sum_{i=0}^{N_3}\xi_i^{(3)}C_{\mu_i}\big(\mathcal{H}_x^{(\mu_i)}(w^j,w^j)-\mathcal{H}_x^{(\mu_i)}(w^{j-1},w^{j-1})\big)\\
&=\frac{1}{2\Delta t}\sum_{i=0}^{N_1}\xi_i^{(1)}\Big(\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\gamma_i}{2}}\hat{w}^{j-1}\parallel_{L^2(\Theta)}^2-\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\gamma_i}{2}}\hat{w}^{j}\parallel_{L^2(\Theta)}^2\Big)\\
&+\frac{1}{2\Delta t}\sum_{i=0}^{N_3}\xi_i^{(3)}\Big(\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\mu_i}{2}}\hat{w}^{j-1}\parallel_{L^2(\Theta)}^2-\parallel {}_{-\infty}\mathbb{D}_x^{\frac{\mu_i}{2}}\hat{w}^{j}\parallel_{L^2(\Theta)}^2\Big),\end{aligned} \tag{46}$$

and in a similar manner, we get

$$\Big(\mathbf{F}_y w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)=\frac{1}{2\Delta t}\sum_{i=0}^{N_2}\xi_i^{(2)}\Big(\parallel {}_{-\infty}\mathbb{D}_y^{\frac{\beta_i}{2}}\hat{w}^{j-1}\parallel_{L^2(\Theta)}^2-\parallel {}_{-\infty}\mathbb{D}_y^{\frac{\beta_i}{2}}\hat{w}^{j}\parallel_{L^2(\Theta)}^2\Big)+\frac{1}{2\Delta t}\sum_{i=0}^{N_4}\xi_i^{(4)}\Big(\parallel {}_{-\infty}\mathbb{D}_y^{\frac{\nu_i}{2}}\hat{w}^{j-1}\parallel_{L^2(\Theta)}^2-\parallel {}_{-\infty}\mathbb{D}_y^{\frac{\nu_i}{2}}\hat{w}^{j}\parallel_{L^2(\Theta)}^2\Big). \tag{47}$$

By the same procedure to calculation of $\Big(\mathbf{F}_x\mathbf{F}_y\varsigma,\varsigma\Big)$, we have

$$\Big(\mathbf{F}_y\varsigma,\varsigma\Big)=2\sum_{i=0}^{N_2}\xi_i^{(2)}\parallel {}_{-\infty}\mathbb{D}_y^{\frac{\beta_{i'}}{2}}\hat{\varsigma}\parallel_{L^2(\Theta)}^2+2\sum_{i=0}^{N_4}\xi_i^{(4)}\parallel {}_{-\infty}\mathbb{D}_y^{\frac{\nu_{i'}}{2}}\hat{\varsigma}\parallel_{L^2(\Theta)}^2, \tag{48}$$

we finally rewrite Eq. (41) as follows

$$\begin{aligned}\Delta\alpha\sum_{k=0}^{2I}d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\vartheta_0^{(\sigma_k)}\Big(\delta_t w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)&+\Delta\alpha\sum_{k=0}^{2I}d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\sum_{l=1}^{j-1}\vartheta_l^{(\sigma_k)}\Big(\delta_t w^{j-l-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)\\
&+\frac{(\Delta t)^2}{4}\Big(\mathbf{F}_x\mathbf{F}_y\delta_t w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)\\
&=\frac{\zeta_0\Delta t}{2}\Big(\mathbf{F}_y\delta_t w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)+\Big(\Big(\mathbf{F}_x+\mathbf{F}_y\Big)w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)\\
&+\Big(h^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big).\end{aligned} \tag{49}$$

We apply the relations (45)-(48) and use the fact $\frac{(\Delta t)^2}{4}\Big(\mathbf{F}_x\mathbf{F}_y\delta_t w^{j-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)\geq 0$ to present

$$\Delta\alpha\sum_{k=0}^{2I}d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\vartheta_0^{(\sigma_k)}\Big|w^j\Big|_{P,Q}^2+\Delta\alpha\sum_{k=0}^{2I}d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\sum_{l=1}^{j-1}\vartheta_l^{(\sigma_k)}\Big(\delta_t w^{j-l-\frac{1}{2}},\delta_t w^{j-\frac{1}{2}}\Big)\leq\Big|w^{j-1}\Big|_{P,Q}^2+2\Delta t\parallel h^{j-\frac{1}{2}}\parallel_{L^2(\Theta)}^2. \tag{50}$$

By summing over j in Eq. (50) from 1 to m and employing Lemma 3.1 and Lemma 4.1 in [40], we derive

$$\parallel w^m\parallel_{P,Q}\leq\mathbb{K}\Big[\Big|w^0\Big|_{P,Q}+\Delta t\sum_{j=0}^{m}\parallel h^{j-\frac{1}{2}}\parallel\Big], \tag{51}$$

and the proof is completed. □

Theorem 3.8. *Assume that $j_1\leq\mathbb{M}+1$ where $\mathbb{M}=\min(N_r)$, $r=1,2,3,4$. Also, let w and w^k be the solutions of Eq. (2) and (19), respectively, where $w\in\mathfrak{H}^3((0,T];\mathfrak{H}^{q_1}(\Theta))$. Then, there exists $\mathbb{K}>0$ independent of k and Δt such that*

$$\parallel w^k-w(x,y,t_k)\parallel_{P,Q}\leq\mathbb{K}\big(\mathcal{O}\big((\Delta t)^2+(\Delta\alpha)^2+(\max(N_r))^{-j_1}\big)\big). \tag{52}$$

Proof. We set $\mathbf{E}^k=w^k-w(x,y,t_k)$ and obtain the error formula as follows

$$\Delta\alpha\sum_{k=0}^{2I}d_k b(\alpha_k)\frac{1}{(\Delta t)^{\sigma_k}}\sum_{l=0}^{j-1}\vartheta_l^{(\sigma_k)}\Big(\delta_t\mathbf{E}^{j-l-\frac{1}{2}},\varsigma\Big)+\frac{(\Delta t)^2}{4}\Big(\mathbf{F}_x\mathbf{F}_y\delta_t\mathbf{E}^{j-\frac{1}{2}},\varsigma\Big)-\frac{\zeta_0\Delta t}{2}\Big(\mathbf{F}_y\delta_t\mathbf{E}^{j-\frac{1}{2}},\varsigma\Big)=\Big(\Big(\mathbf{F}_x+\mathbf{F}_y\Big)\mathbf{E}^{j-\frac{1}{2}},\varsigma\Big)+\Big(\Xi_1^m,\varsigma\Big),\ \forall\varsigma\in\mathbb{S}_N, \tag{53}$$

where

$$\Xi_1^m = \mathcal{O}\big((\Delta t)^2 + (\Delta\alpha)^2 + (\max(N_r))^{-j_1}\big).$$

We now use Theorem 3.7 and consider $\varsigma = \delta_t \mathbf{E}^{j-\frac{1}{2}}$ to get

$$\| w^k - w(x,y,t_k) \|_{P,Q} \leq \mathbb{K}\big((\Delta t)^2 + (\Delta\alpha)^2 + (\max(N_r))^{-j_1}\big). \tag{54}$$

Therefore, the proof is completed. □

Theorem 3.9. *Suppose that w_N^m is the solution of Eq. (21). Then, there exists $\mathbb{K} > 0$, independent of m and Δt, such that*

$$\| w_N^m \|_{P,Q} \leq \mathbb{K}\Big[\Big|w_N^0\Big|_{P,Q} + \Delta t \sum_{j=1}^{m} \| h^{j-\frac{1}{2}} \| \Big]. \tag{55}$$

Proof. In Theorem 3.7, we consider $\varsigma = \delta_t w_N^{j-\frac{1}{2}}$ and apply the same process. As a result, Eq. (55) is obtained. □

Theorem 3.10. *Assume that $j_1 \leq \mathbb{M} + 1$ and $j_1 \geq 1$. Also, let w and w_N^k be the solutions of Eqs. (2) and (21), respectively, where $w \in \mathfrak{H}^3((0,T];\mathfrak{H}^{q_1}(\Theta))$. Then, there exists $\mathbb{K} > 0$, independent of k and Δt, such that*

$$\| w_N^k - w(x,y,t_k) \|_{P,Q} \leq \mathbb{K}\Big((\Delta t)^2 + (\Delta\alpha)^2 + (\mathbb{M})^{-j_1} + \sum_{i=0}^{N_1} \xi_i^{(1)} N^{\frac{\gamma_i}{2}-q_1} + \sum_{i=0}^{N_2} \xi_i^{(2)} N^{\frac{\beta_i}{2}-q_1} + \sum_{i=0}^{N_3} \xi_i^{(3)} N^{\frac{\mu_i}{2}-q_1} + \sum_{i=0}^{N_4} \xi_i^{(4)} N^{\frac{\nu_i}{2}-q_1}\Big). \tag{56}$$

Proof. Consider $\tilde{w} = \Lambda_N^{P,Q} w$, $\lambda = w - \tilde{w}$ and $E = \tilde{w} - w_N$. We can display the error formula as follows

$$\begin{aligned}
\Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{j-1} \vartheta_l^{(\sigma_k)} \Big(\delta_t E^{j-l-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) + \frac{(\Delta t)^2}{4}\Big(\mathbf{F}_x\mathbf{F}_y\delta_t E^{j-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) - \frac{\zeta_0 \Delta t}{2}\Big(\mathbf{F}_y \delta_t E^{j-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) \\
= \Big(\Big(\mathbf{F}_x + \mathbf{F}_y\Big) E^{j-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) \\
- \Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{j-1} \vartheta_l^{(\sigma_k)} \Big(\delta_t \lambda^{j-l-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) \\
- \frac{(\Delta t)^2}{4}\Big(\mathbf{F}_x\mathbf{F}_y\delta_t \lambda^{j-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) + \frac{\zeta_0 \Delta t}{2}\Big(\mathbf{F}_y \delta_t \lambda^{j-\frac{1}{2}}, \delta_t E^{j-\frac{1}{2}}\Big) \\
+ \mathcal{O}\big((\Delta t)^2 + (\Delta\alpha)^2 + (\mathbb{M})^{-j_1}\big).
\end{aligned} \tag{57}$$

To calculate the error, we only need to calculate the following expression

$$\Big|E^0\Big|_{P,Q} + \| \mathcal{O}\big((\Delta t)^2 + (\Delta\alpha)^2 + (\mathbb{M})^{-j_1}\big) - \Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{j-1} \vartheta_l^{(\sigma_k)} \delta_t \lambda^{j-l-\frac{1}{2}} - \frac{(\Delta t)^2}{4}\mathbf{F}_x\mathbf{F}_y\delta_t\lambda^{j-\frac{1}{2}} + \frac{\zeta_0\Delta t}{2}\mathbf{F}_y\delta_t\lambda^{j-\frac{1}{2}} \|, \ j = 1,2,\ldots,M,$$

where E^0 is the initial error. For E^0, we obtain the following inequality

$$\begin{aligned}
\Big|E^0\Big|_{P,Q} &\leq \| E^0 \|_{P,Q} = \| \Lambda_N^{P,Q} w_0 - \Lambda_N^{1,0} w_0 \|_{P,Q} \\
&\leq \| \Lambda_N^{P,Q} w_0 - w_0 \|_{P,Q} + \| w_0 - \Lambda_N^{1,0} w_0 \|_{P,Q} \\
&\leq \mathbb{K}\Big(\sum_{i=0}^{N_1} \xi_i^{(1)} N^{\frac{\gamma_i}{2}-q_1} \\
&+ \sum_{i=0}^{N_2} \xi_i^{(2)} N^{\frac{\beta_i}{2}-q_1} + \sum_{i=0}^{N_3} \xi_i^{(3)} N^{\frac{\mu_i}{2}-q_1} + \sum_{i=0}^{N_4} \xi_i^{(4)} N^{\frac{\nu_i}{2}-q_1}\Big).
\end{aligned} \tag{58}$$

Also, we have

$$\| \frac{(\Delta t)^2}{4}\mathbf{F}_x\mathbf{F}_y\delta_t\lambda^{j-\frac{1}{2}} \| \leq \mathbb{K}(\Delta t)^2, \tag{59}$$

and

$$\| \Delta\alpha \sum_{k=0}^{2I} d_k b(\alpha_k) \frac{1}{(\Delta t)^{\sigma_k}} \sum_{l=0}^{j-1} \vartheta_l^{(\sigma_k)} \delta_t \lambda^{j-l-\frac{1}{2}} \| \leq \mathbb{K}\Big|\delta_t\lambda^{j-\frac{1}{2}}\Big|_{P,Q} \leq \mathbb{K}\Big(\sum_{i=0}^{N_1} \xi_i^{(1)} N^{\frac{\gamma_i}{2}-q_1} + \sum_{i=0}^{N_2} \xi_i^{(2)} N^{\frac{\beta_i}{2}-q_1} + \sum_{i=0}^{N_3} \xi_i^{(3)} N^{\frac{\mu_i}{2}-q_1} + \sum_{i=0}^{N_4} \xi_i^{(4)} N^{\frac{\nu_i}{2}-q_1}\Big). \tag{60}$$

By combining (58)-(60) and applying Lemma 3.3, we have

$$\| w_N^k - w(x,y,t_k) \|_{P,Q} \leq \mathbb{K}\Big((\Delta t)^2 + (\Delta\alpha)^2 + (\mathbb{M})^{-j_1} + \sum_{i=0}^{N_1} \xi_i^{(1)} N^{\frac{\gamma_i}{2}-q_1} + \sum_{i=0}^{N_2} \xi_i^{(2)} N^{\frac{\beta_i}{2}-q_1} + \sum_{i=0}^{N_3} \xi_i^{(3)} N^{\frac{\mu_i}{2}-q_1} + \sum_{i=0}^{N_4} \xi_i^{(4)} N^{\frac{\nu_i}{2}-q_1}\Big). \tag{61}$$

The proof of this theorem is completed. □

4. Numerical examples

In this section, we discuss about some numerical examples to confirm and show the efficiency of the proposed method. Moreover, in this section, we define the following symbols for the convergence orders (CO) and the absolute error (AE) in the L^2-norm formulas

$$\mathrm{CO} = \begin{cases} \dfrac{\log\left(\frac{\|AE(\Delta t_1)\|}{\|AE(\Delta t_2)\|}\right)}{\log\left(\frac{\Delta t_1}{\Delta t_2}\right)}, & \text{in time}, \\ \dfrac{\log\left(\frac{\|AE(N_1)\|}{\|AE(N_2)\|}\right)}{\log\left(\frac{N_2}{N_1}\right)}, & \text{in space}, \end{cases}$$
$$\mathrm{AE} = \| w_{\text{exact solution}} - w_N^m \|, \tag{62}$$

where w_N^m is the approximate solution of Eq. (2).

Example 4.1. We consider the two-dimensional space-time distributed-order fractional diffusion-wave equations with the Riesz space fractional derivative as follows

$$\int_1^2 \frac{8}{15\sqrt{\pi}}\Gamma(\frac{7}{2}-\alpha)^C\mathfrak{D}_t^\alpha w(x,y,t)d\alpha = \int_1^2 Q_1(\gamma)\frac{\partial^\gamma w(x,y,t)}{\partial|x|^\gamma}d\gamma + \int_1^2 Q_1(\gamma)\frac{\partial^\gamma w(x,y,t)}{\partial|y|^\gamma}d\gamma$$
$$+ \int_0^1 Q_3(\mu)\frac{\partial^\mu w(x,y,t)}{\partial|x|^\mu}d\mu + \int_0^1 Q_3(\mu)\frac{\partial^\mu w(x,y,t)}{\partial|y|^\mu}d\mu + h(x,y,t),$$

with the initial and boundary conditions

$$w(x,y,t)=0,\ (x,y,t)\in\partial\Theta\times(0,1],$$
$$w(x,y,0)=\big(xy(1-x)(1-y)\big)^2,\ w_t(x,y,0)=0,\ (x,y)\in\Theta=(0,1)\times(0,1),$$

where

$$Q_1(\gamma)=-2\Gamma(5-\gamma)\cos(\frac{\pi\gamma}{2}),\ Q_3(\mu)=-2\Gamma(5-\mu)\cos(\frac{\pi\mu}{2}),$$
$$h(x,y,t)=\frac{\sqrt{t}(t-1)}{\ln t}\big(xy(1-x)(1-y)\big)^2-(t^2\sqrt{t}+1)(y(1-y))^2\Big[f_1(x)+f_1(1-x)+f_2(x)$$
$$+f_2(1-x)\Big]-(t^2\sqrt{t}+1)(x(1-x))^2\Big[f_1(y)+f_1(1-y)+f_2(y)$$
$$+f_2(1-y)\Big],$$
$$f_1(u)=\Gamma(5)\frac{u^3-u^2}{\ln u}-2\Gamma(4)\Big(\frac{3u^2-2u}{\ln u}-\frac{u^2-u}{(\ln u)^2}\Big)$$
$$+\frac{\Gamma(3)}{\ln u}\Big(6u-2-\frac{5u}{\ln u}+\frac{3}{\ln u}+\frac{2u}{(\ln u)^2}-\frac{2}{(\ln u)^2}\Big)$$
$$f_2(u)=\Gamma(5)\frac{u^4-u^3}{\ln u}-2\Gamma(4)\Big(\frac{4u^3-3u^2}{\ln u}-\frac{u^3-u^2}{(\ln u)^2}\Big)$$
$$+\frac{\Gamma(3)}{\ln u}\Big(12u^2-6u-\frac{1}{\ln u}\Big(7u^2-5u-\frac{2u^2}{(\ln u)}+\frac{2u}{(\ln u)}\Big)\Big).$$

The exact solution for this problem is $w(x,y,t)=(t^2\sqrt{t}+1)\big(xy(1-x)(1-y)\big)^2$. We solve this problem by considering the values $\Delta t=0.01$, $N=50$ and $\min(N_r)=10$, $r=1,2,3,4$. Fig. 1 displays the plots of the approximate solution for this problem. Plots of the absolute error via different values of t have been displayed in Fig. 2. Table 1 reports the comparisons of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO at time $t=1$ with the change of Δt, when $N=32$ and $\min(N_r)=10$. From the Table 1, we can see that CO with respect to time is the second order. Table 2 reports the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO at time $t=1$ when $\Delta t=0.001$ and $\min(N_r)=10$.

Example 4.2. We consider the two-dimensional space-time distributed-order fractional diffusion-wave equations with the Riesz space fractional derivative as follows

$$\int_1^2 \Gamma(5-\alpha)^C\mathfrak{D}_t^\alpha w(x,y,t)d\alpha = \int_1^2 e^{-2\gamma}\frac{\partial^\gamma w(x,y,t)}{\partial|x|^\gamma}d\gamma + \int_1^2 e^{-2\gamma}\frac{\partial^\gamma w(x,y,t)}{\partial|y|^\gamma}d\gamma$$
$$+ \int_0^1 e^{-2\mu}\frac{\partial^\mu w(x,y,t)}{\partial|x|^\mu}d\mu + \int_0^1 e^{-2\mu}\frac{\partial^\mu w(x,y,t)}{\partial|y|^\mu}d\mu + 1,$$

with

$$w(x,y,t)=0,\ (x,y,t)\in\partial\Theta\times(0,10],$$
$$w(x,y,0)=e^{-10[(x-0.5)+(y-0.5)]},\ w_t(x,y,0)=0,\ (x,y)\in\Theta=(0,1)\times(0,1).$$

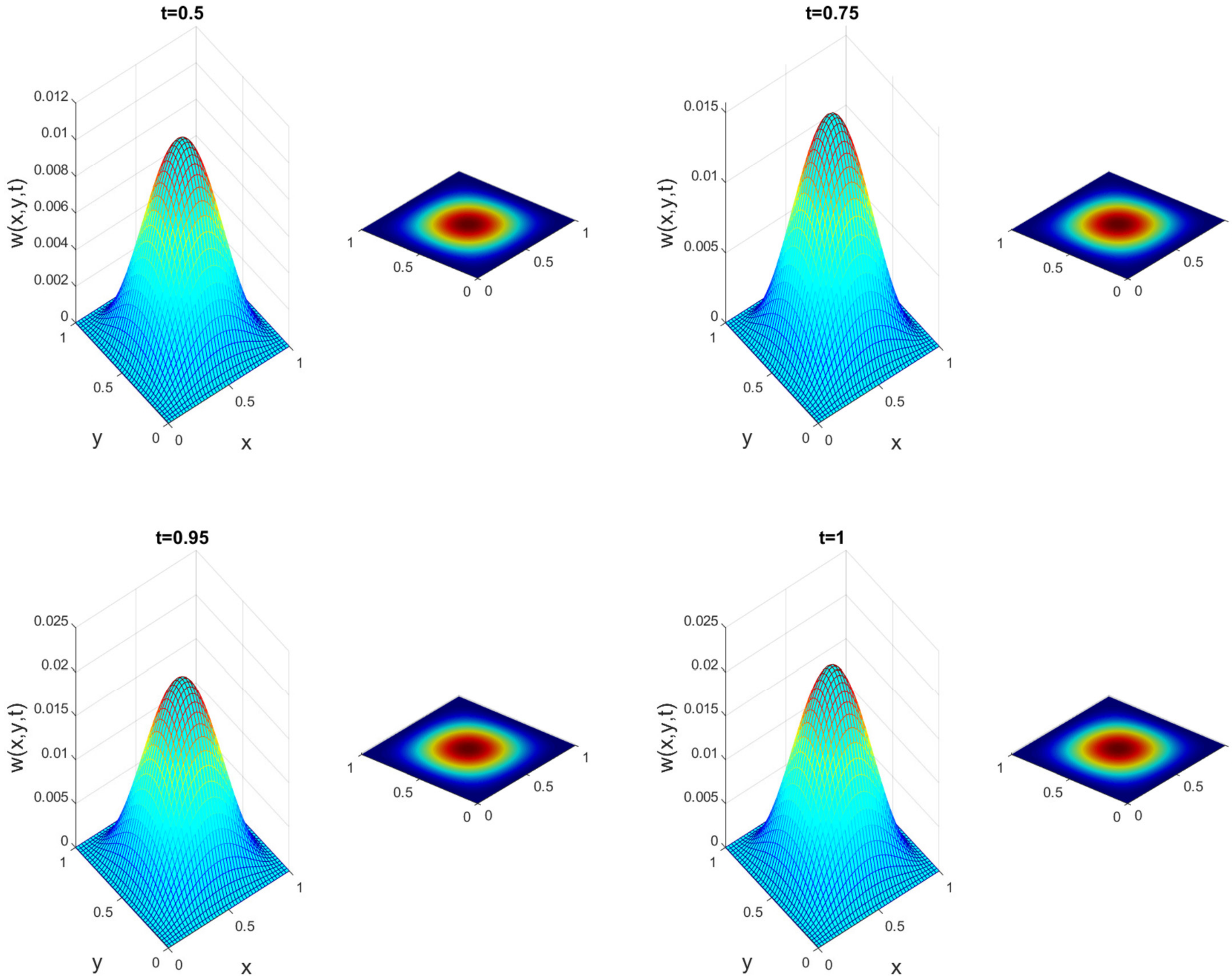

Fig. 1. Graph of the numerical solution for Example 4.1 with different kinds of t.

To compute the error approximate by using of the proposed numerical method, we estimate the error by

$$error(\Delta t) = \| w(x,y,t,0.00005) - w(x,y,t,\Delta t) \|,$$

or

$$error(N) = \| w(x,y,t,1024) - w(x,y,t,N) \|,$$

when $w(x,y,t)$ shows the numerical solution of this problem. The numerical results of the proposed numerical method have been illustrated in Fig. 3 for the different values of t. From Fig. 3, we see that the maximum peak always occurs at the center of the domain. Table 3 reports the comparison of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO with the change of Δt for this problem by using the proposed numerical method when $N = 50$. From Table 3, we can see that CO with respect to time is the second order. Table 4 reports the comparison of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO with the change of N when fixing $\Delta t = 0.00001$. It is obvious that the L^2-error has a higher computational accuracy than the $\mathfrak{H}^{P,Q}$-error.

5. Conclusions

In this manuscript, we use the difference-Legendre spectral method to solve the two-dimensional space-time distributed-order fractional diffusion-wave equations with the Riesz space fractional derivative. The numerical solution of the introduced equations was studied and the associated convergence and stability analyses were clarified. We obtain a second-order convergence in time and show some numerical examples to display the effectiveness of suggested numerical methods.

Data availability

No data was used for the research described in the article.

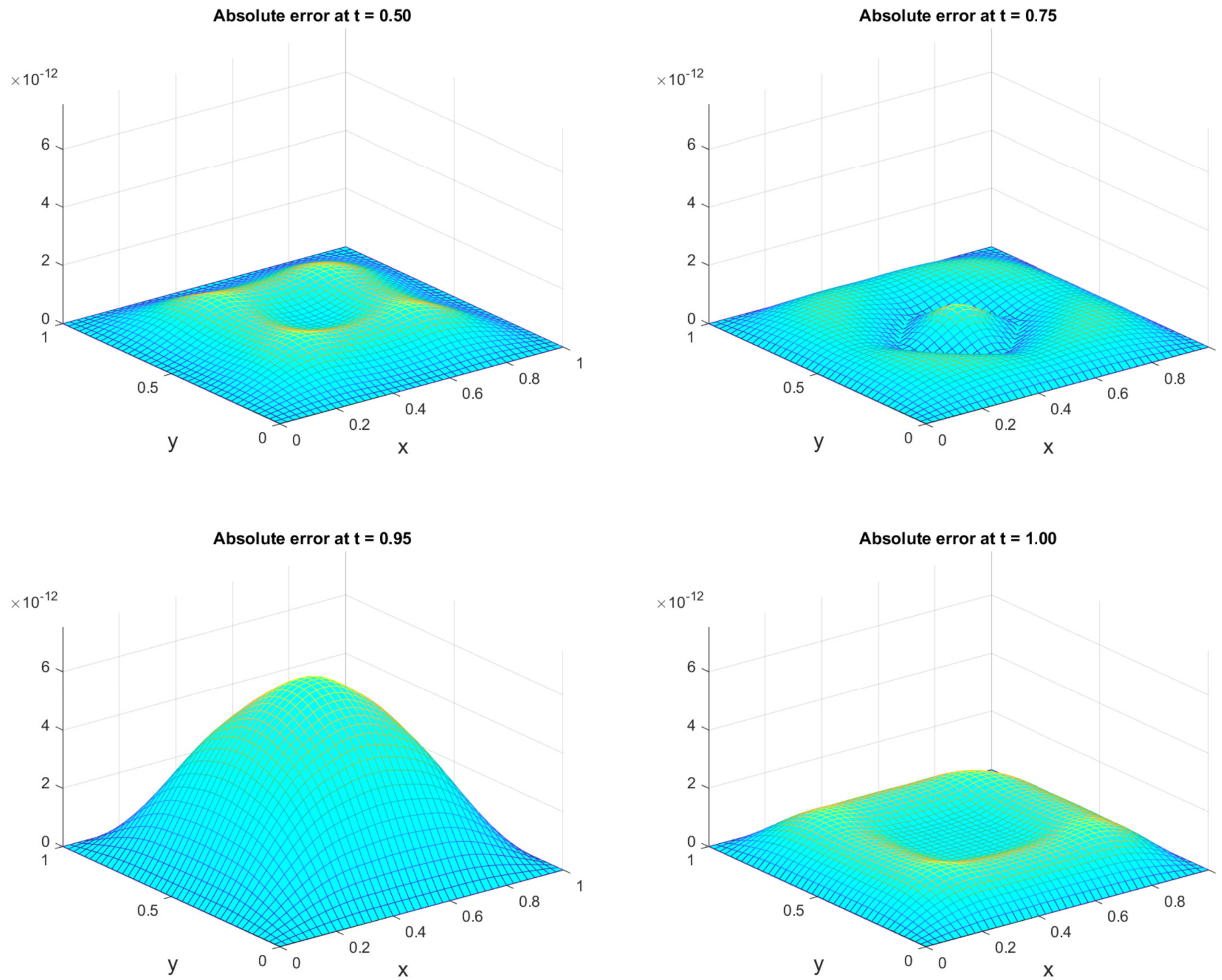

Fig. 2. Graph of the absolute error function for Example 4.1 with different kinds of t.

Table 1
Comparison of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO with the change of Δt for Example 4.1.

Δt	L^2-errors	CO	$\mathfrak{H}^{P,Q}$-errors	CO
$\frac{1}{20}$	$5.7663e-08$	–	$5.7523e-07$	–
$\frac{1}{40}$	$2.5330e-09$	1.9980	$2.4110e-08$	1.9797
$\frac{1}{80}$	$1.1717e-11$	1.9998	$1.1505e-10$	1.9989
$\frac{1}{160}$	$6.6058e-14$	2.0001	$7.3037e-13$	1.9995

Table 2
Comparison of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO with the change of N for Example 4.1.

N	L^2-errors	CO	$\mathfrak{H}^{P,Q}$-errors	CO
8	$3.4794e-08$	–	$1.8597e-07$	–
16	$7.0509e-10$	1.7256	$3.7455e-9$	1.5036
32	$3.3293e-11$	3.9924	$1.7746e-10$	3.7704
64	$1.4740e-13$	5.5294	$1.8940e-12$	4.3074

Acknowledgements

I would like to express my special appreciation and thanks to Professor Alireza Ansari for his advice on concepts of fractional differential equations of distributed-orders.

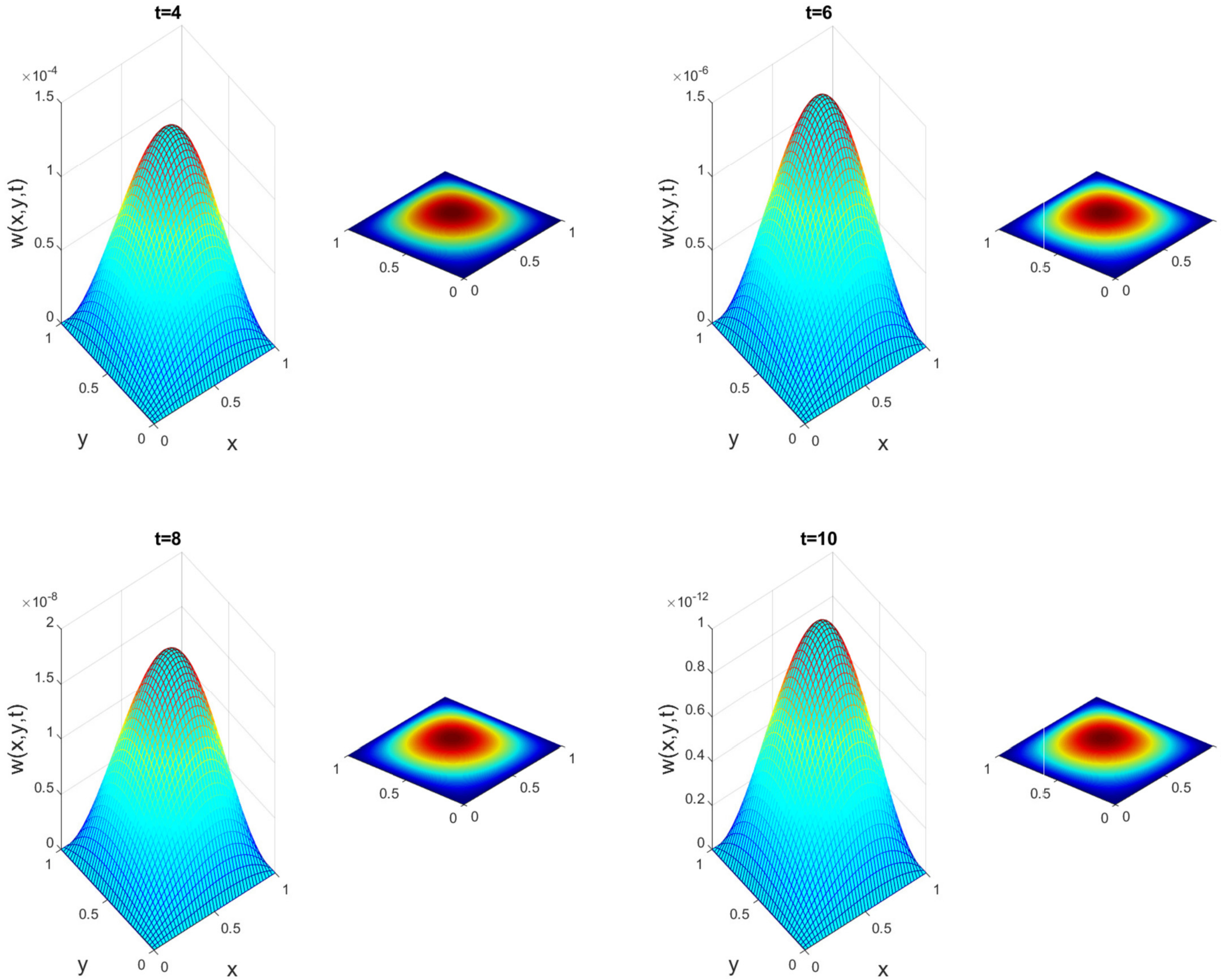

Fig. 3. Graph of the numerical solution for Example 4.2 with different kinds of t.

Table 3
Comparison of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO with the change of Δt for Example 4.2.

Δt	L^2-errors	CO	$\mathfrak{H}^{P,Q}$-errors	CO
$\frac{1}{80}$	$3.6738e-10$	–	$7.6791e-10$	–
$\frac{1}{160}$	$7.4801e-11$	1.8862	$3.2101e-10$	1.8552
$\frac{1}{320}$	$4.7301e-12$	2.0020	$7.8113e-12$	1.8820
$\frac{1}{640}$	$3.2335e-13$	2.0065	$3.0123e-12$	2.0013

Table 4
Comparison of the L^2-errors, $\mathfrak{H}^{P,Q}$-errors and CO with the change of N for Example 4.2.

N	L^2-errors	CO	$\mathfrak{H}^{P,Q}$-errors	CO
16	$4.4391e-06$	–	$3.1646e-05$	–
32	$3.5669e-07$	1.8345	$2.4459e-06$	1.3345
64	$9.1265e-09$	3.8856	$9.0153e-08$	2.8056
128	$5.3411e-11$	5.4455	$6.5771e-10$	4.4455

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